Mechanics, Heat, Oscillations and Waves
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Lecture – 30
Damped Simple Harmonic Motion

Next, let us take up to the example of a Damped Simple Harmonic Motion and the idea is the following, if I look at a simple harmonic oscillator, such as a mass and a spring connected to the mass and the simple harmonic oscillation of this mass.

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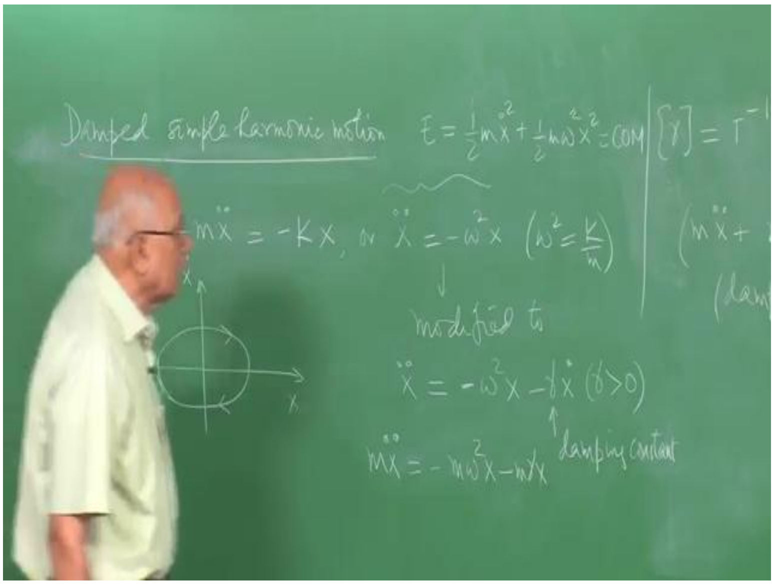

You have seen that the equation of motion of this mass Newton's equation is m x double dot equal to minus k x, where k is the spring constant or x double dot equal to minus omega squared x, where omega squared is k over m here. Now, this motion goes on forever for any initial conditions, because there is no mechanism by which energy is dissipated. The energy of the spring of the system is constant for any given initial conditions.

And if you recall the face trajectory in this case, in the x dot explain this face trajectory was in the form of ellipses for any given energy travels in the clock wise sense. There is no damping or dissipation in the system, this oscillator goes on forever and that of course, is unrealistic in any physical oscillator, they will always be a mechanism by which some loss of energy is incurred, the system comes to a haled either by friction or by some other damping mechanism. And we need to in cooperate that in our understanding of simple harmonic motion.

Basically, what happens, when you include the effect of such dissipation on this conservative motion? Now, this depends on the model, this depends on what kind of a friction we have, what kind of what is the precise mechanism by which this friction a dissipates energy and so on. But, a very simple and standard model, which is applicable in a large number of physical situations, is the following.

It is exactly what happens, when you drop an object like a ball bearing inside a dense liquid, what happens is due to the viscosity of the liquid, the particles velocity get slow down, there is friction on this object, because of viscosity. And there is what is known is viscous drag or viscous damping, which causes slowing down and dissipation of energy. Now, just as when you walk through a crowd, the faster you try to go in a particular direction, the more the returning force, because the more the number of collisions from the front in exactly the same way.

The discuss damping in a simplest form is a drag force, which is opposite to the direction in which the velocity is directed or opposite to the direction of motion and is proportional to the velocity itself. So, the very simplest model that we can put in for damping here is to modify this equation, modify to x double dot equal to something which is proportional to the velocity.

Of course, there is this restoring term still this spring force still there, but there is also a term, which is in the direction opposite to the direction of the velocity. So, it is proportional to minus x dot times some constant. So, minus some gamma x dot and this gamma is the damping constant of the friction constant. Notice from this that gamma is actually physical quantity of dimensions the same as that are frequency t inverse.

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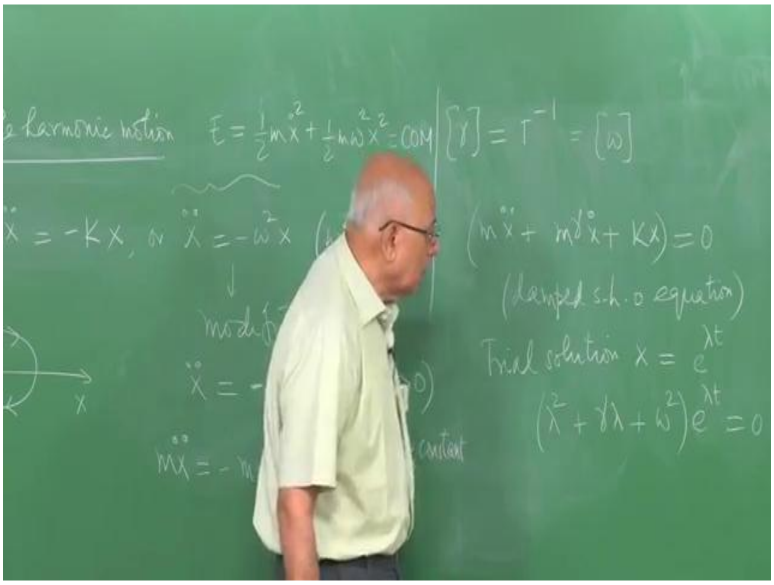

So, the physical dimensions of gamma are those of inverse time, the same as those of frequency. So, what have to done is to really take this equation and rewrite it in the form mass times the acceleration equal to minus m omega squared x minus m gamma x and canceled out the m and also both sides in order to get this equation. So, this standard equation of the form m x double dot plus m gamma x dot plus k x., if you like equal to 0 is the standard equation for the damped simple harmonic oscillator in one dimension.

Where, this is the damping constant and that is the spring constant and that is the general equation. So, anything which is of this form, any second order differential equation of this form would be that of the damped simple harmonic oscillator. So, this is the damped s h o equation and a little later, we will look at an electrical analogy for this mechanical damped equation.

But, before that, let us try to look at this solutions of this equation here, earlier I mention that this equation must entirely equivalent to saying the total energy E was equal to 1 half a p squared or x 1 half m x dot squared plus 1 half m omega squared x squared equal to a constant of the motion; that is no longer true. So, this was a constant of the motion and that is what let to this ellipse as the phase trajectory, such an expression for a constant energy is not available in this case.

This equation is not consistent with saying that the energy given by this expression is a constant. In fact, energy is not conserved in this problem as we will see in a minute, I have to mention very carefully that gamma is greater than 0. Because, the friction is opposed to the motion to the direction of the velocity, so the minus sign and then, a positive constant times x dot here, I have choose in the constant to be gamma.

Actually, I have return m gamma in the equation of motion taking out in m simplify for convenient, so that gamma as a same physical dimensions is omega and the idea is we will see shortly will be to compare the relative magnitudes of gamma and omega. That it control the nature of the motion here. How does one solve equation of this kind? Well, there are standard methods for solving such a second order differential equation in elementary calculus.

But, what we need to do here is to write down the solution based almost on inspection, it was clear here that the solutions, where E to the plus or minus i omega t. So, some kind of exponential and one tries the same thing here, because what happens when you have a linear differential equation and this is linear. Because, it is a first power of x here, first power of x dot first power of x double dot is to notice that if you take quantity like E to the lambda t, well lambda is a constant and you differentiated it once, you bring a lambda down, you differentiated the second time you bring another lambda down. So, that the operation of differentiation becomes equivalent to the operation of multiplication, when it acts on E to the lambda t. So, that suggest immediately that want tries a trial solution x equal to some constant. Because, this is a linear equation multiplicand constant with fix later on by initial conditions equal to E to the power lambda t.

Well, lambda is as yet undetermined, if you do that and in put it in here as I said each time, you differentiate with respect to time, you being down a lambda. So, it says this equation simply says lambda squared plus gamma lambda plus omega squared equal to 0. Because, this whole thing is multiplied by E to the lambda t and that must be equal to 0.

I have now taken out them and return k over m is omega squared. So, that you have two constants gamma and omega squared in the problem, lambda is unknown as yet, but it has to satisfy this condition here for all t. Since, E to the lambda t is not identically equal to 0, this can only we have done, if this bracket vanishes.

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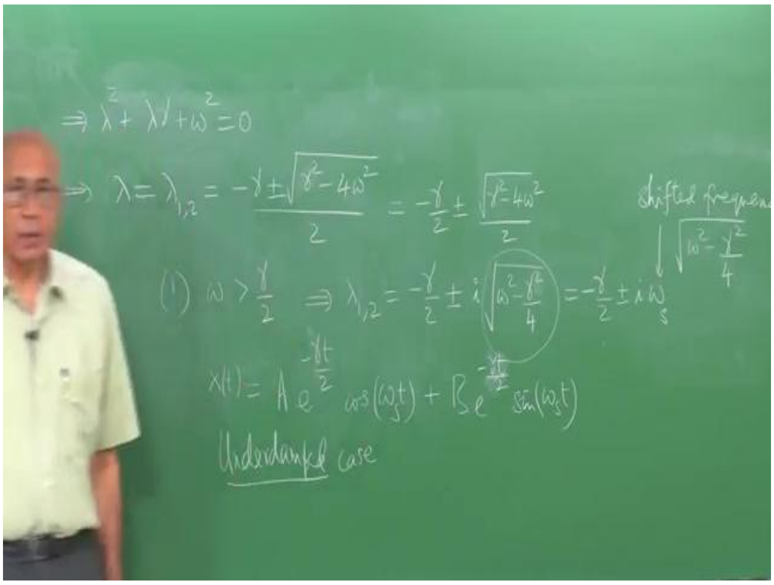

So, this implies that lambda must be such that lambda squared plus lambda gamma which in turn implies that you have two possible values lambda can only be equal to lambda 1 or 2 which I given by minus gamma plus or minus square root of omega gamma squared minus 4 omega squared over 2 these are the two possible roots of this quadratic. So, it says that a trial solution of this exponential form is admissible for this equation provided lambda is not arbitrary, but can take on either this value or that value one of the two roots here, one above obvious values here.

And since this is a linear equation, this equation is a linear equation if I find one solution and I find another solution, any linear combination of the two is also a solution. So, the most general solution of this equation this would be a form x of t equal to some constant times e to the lambda 1 t let us call it c 1 for the moment plus some other constant e to the lambda 2 t that is the most general solution, where the constants c 1 and c 2 have yet to be determine.

But, the exponents the coefficients lambda 1 and lambda 2 are determine in terms of the parameters in the problem, namely gamma and omega by these formulas here, these expressions here these are the only these are the only possibilities for the general solution. But, as soon as you see this let us rewrite this little bit more simply, so let us rewrite this as plus or minus minus pi, it is immediately clear that three possibilities arise, three distinct possibilities arise depending on the relative strengths of gamma and omega.

So, the first thing that arises the first case one would be the case where this number is bigger than that number, which means omega is bigger than gamma over 2 and that physically corresponds to the case where you essentially have an undammed simple harmonic oscillator with natural frequency omega. And the friction constant is small compared to the natural frequency or the natural frequency is bigger than half the friction constant here that is all that is needed omega must be greater than lambda over 2.

So, 4 omega squared is greater than gamma squared which will immediately mean that lambda 1 comma 2 is minus gamma over 2 plus or minus i times omega squared minus gamma squared over 4 and remember this is a positive number. So, we have a pair of complex conjugates roots with the real part which is non zero and then imaginary part which is non zero. And let us give name to this quantity here, this is the equal to minus gamma over 2 plus or minus i times omega shifted. So, this is the shifted frequency or shifted angle of frequency omega squared minus gamma squared over 4 square root.

So, we still have the possibility of oscillatory solutions, but the frequency shifted and there is a minus gamma over 2 sitting outside. So, this tells you that the solution is exponentially damped. So, this would imply that x of t in general is of the form A e to the minus gamma t over 2 that corresponds to this part of the solution times e to the plus or minus i omega s which I could as well write as either cos omega s t plus b e to the minus lambda t gamma t over 2 sin omega s t.

So, I have solution which are oscillatory, but with the shifted frequency not the original natural frequency, but this combination omega squared minus gamma squared over 4 is new frequency and there is an exponentially damping factor here, this is called the under damped case. What does it mean? It means that the solutions, if you plot the solutions for instance let us look at initial conditions, where x of 0 is A and x dot of 0 is 0. So, that this part goes away completely.

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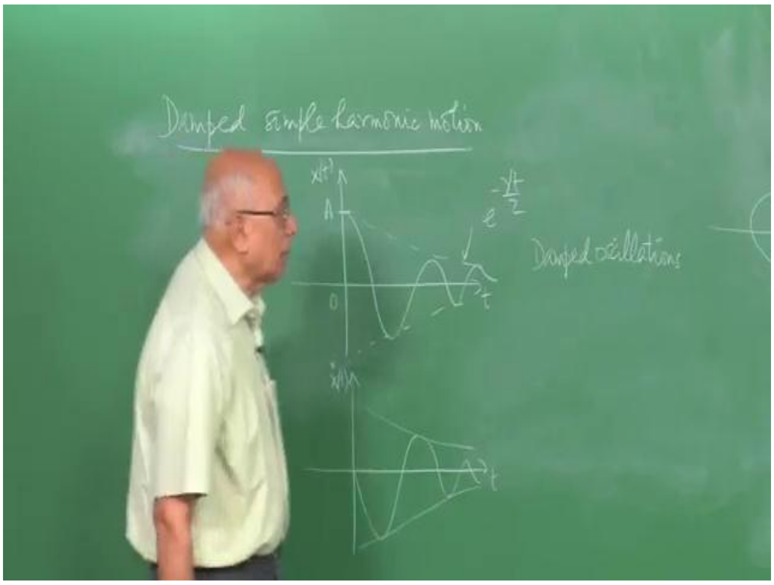

And then you are left with a solution is which looks like this here is t, here is x of t a t equal to 0 it starts at the point A and then it osculates with a frequency omega sub s angular frequency omega sub s, but instead of being just a cosine curve it is now damp by this exponential. So, this looks like and this damping here this function the envelope function is e to the minus gamma t over 2 and this is minus this everything. So, it leads to damped oscillations

What does x dot do in such a case well that is easily plotted to here is t, here is x dot of t and that is starts with 0, because if x starts with the cosine and there is no sin here, then if I differentiate this once I get sin t here omega s t and that is 0 at t equal to 0 and then it is negative. So, this thing it is starts at a moves backwards, so you here you started this point and then it does this. So, again there is a damping here in this fashion and both the velocity and the position tends toward 0 in this case. What happens in the face plain?

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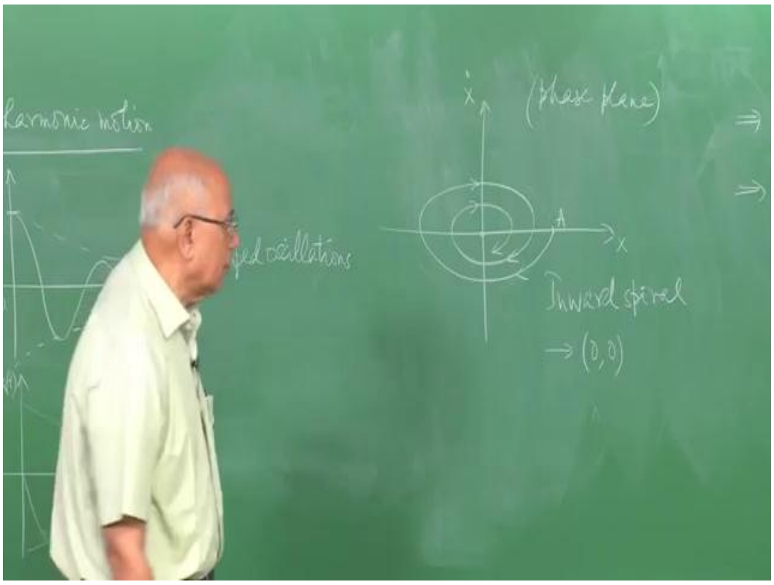

So, face plot is what revels to us what the true nature of this solution is here is x and here is x dot and this is the face plain, in the options of damping we would a had ellipses of this kind. But, now the system starts at A, but this because of the damping what it does is to decrease does not go quite back to A and then keeps doing this and then in inwards spiral it is spiral towards x dot equal to 0 and x equal to 0.

The fact that x changes sin we decreasing amplitude is precisely what makes the spiral there and that is reflected here x changes sin goes back and forth, but it decreasing amplitudes. So, this is not a ellipses; obviously, it is a spiral of some kind it is an inwards spiral tending towards 0, 0 it tends towards the origin in the x plane and x dot plain and that is under damped simple harmonic motion. There are cosines and sins, the frequency shifted from the original un shifted frequency of omega to omega squared minus gamma squared over 4 square root and there is an overall exponential damping factor with the characteristic time which is 2 over gamma and that is exactly what these exponential certain us.

So, this is the under damped case in which you have oscillatory behave an oscillatory approach towards the equilibrium point of 0 velocity and 0 position. The next case which is not quite so interesting is a case when this bracket vanishes exactly when the square root vanishes.

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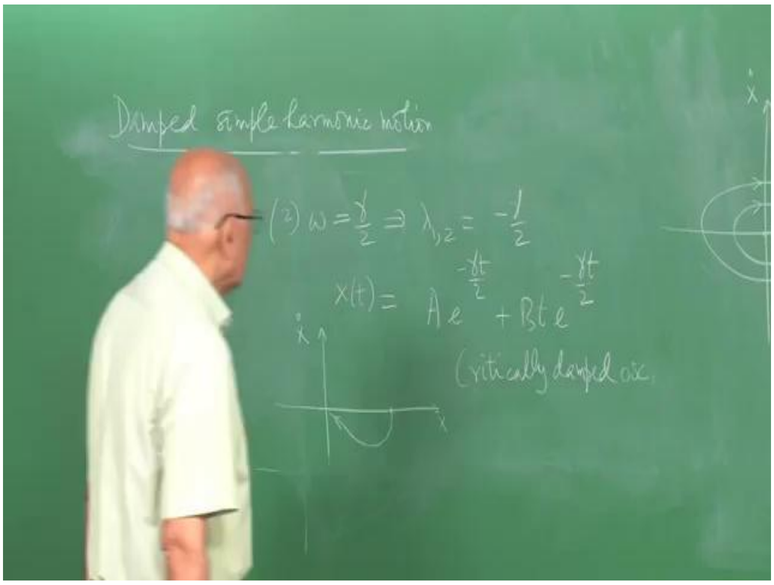

So, we have a second case the omega is exactly equal to gamma over 2 then this bracket vanishes the square root and you are left with lambda 1 which implies lambda 1 comma 2 about the same about equal to gamma over minus gamma over 2. The general solution in this case; obviously, cannot we have the form A to the minus lambda 1 t, e to the lambda 1 t plus e 2 to the lambda 2 t for the simple reason that they both the same solution, general solution is of the form x of t is A to the minus gamma t over 2 plus B t e to the minus gamma t over 2. I shall not go into how this is arises it is again a straight forward consequence of solving this equation using a proper procedure.

But, that fact, that you have a t here, next the solution linearly independent from this solution that two different functions of t. On the other hand, that the same common exponential factor one each of them and as t becomes very large this is not going to dominate over that the exponential dominates goes to at 0 exponentially fast although this arises this takes care of it and tends go down to 0.

So, this is again an exponentially damped solution and what happens here in the face plane is simply that you have an approach you start at x here, x dot here, you start here it goes up trying to become a spiral and then slowly tends towards ((Refer Time: 20:33)) it does not cross the x axis never becomes negative, because is this thing here for in initial condition such that A is positive and B is 0 is always going to be just an exponential damped quantity which is got the same sin. So, x remains positive, but x dot slowly tends towards 0 both of them exponential fast this the critically damped case oscillator and the third case is when omega becomes less than gamma over 2.

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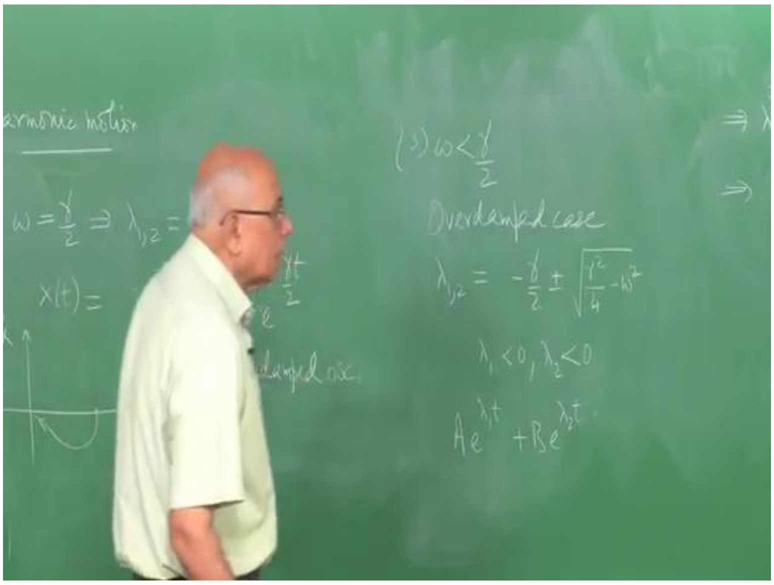

We have already saw that this was will under damped case this one is the so called over damped case and what happens to the lambda 1 and 2. In this case, lambda 1 comma 2 are both given by just that factor there minus gamma over 2 plus or minus square root of gamma squared minus 4 omega squared or rather m squared over write it as square root of gamma squared over 4 minus omega square. And please note that both of them and negative. So, lambda 1 is negative, lambda 2 is negative both are real and both of them negative.

And the general solution which is of the form A e to the lambda 1 t plus B e to the lambda 2 t is a super position of two damped exponentials one of which must faster damping exponential, the one in both minus sign, where as the one of the plus sign does not decay as fast. Because, it is not shown negative, the picture looks pretty much like this in fact, it looks like even more damped to it will damp out even more rapidly in this case and we not going to be concern in much more this.

Because, it just says monotonically these functions x will just decay in time monotonically s t increases and not very interesting as for as we are concern, the under damped case where still have oscillations when you have a small amount of damping is physically very often, the most interesting case of all. Now, let see whether we can look at another physical system which displace exactly the same sort of behavior and this has to do with the electrical analogy which I made last time to the undamped oscillator.

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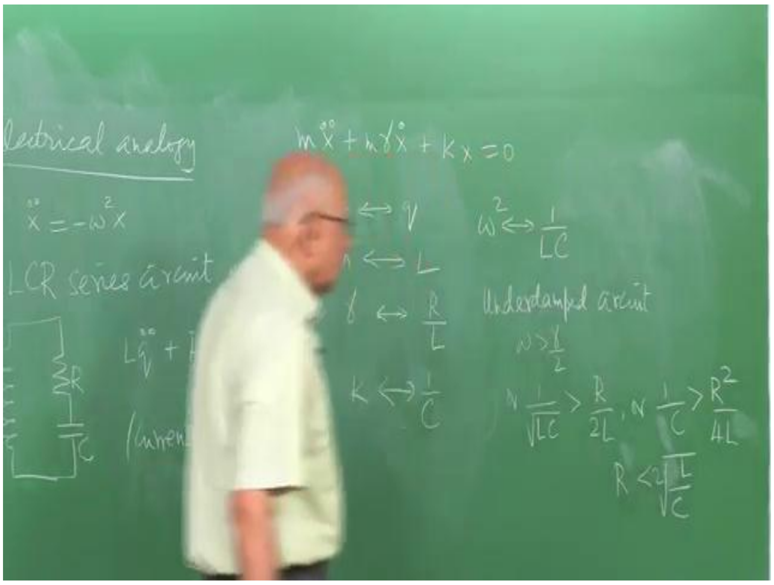

If you recall we looked at electrical analogy, if you recall for the un damped oscillator x double dot equal to minus omega squared x we also had and L C circuit which look like this you had an L here and C here and the charge on this capacitor q behave like L q double dot plus q over C was equal to 0. Since, there is no apply voltage on the circuit and this L C parallel circuit has exactly the same behavior as this equation x q double dot equal to minus omega squared q and omega is 1 over square root of L C.

We now need to introduce a damping, so that the equation looks like a term with q dot present unit. And of course, we know what q dot is it the current in the circuit and we need something the voltage across which is proportional to the current unit and that of course, is the resistor immediately. So, if I have and LCR circuit and this is the direct analog of the damped oscillator, so it is called LCR series circuit, we have a inductor or resistor R and a capacitor C.

Then the equation obit in this circuit by the initial charge on the capacitor will change as a function of time according to L q double dot plus R times q dot and this is the current and that is of course, a rate of change of the current which is what is giving you by the back EMF of the cross inductive by faradays law of induction plus q c equal to 0. Now, this is the exact mechanical analog of the equation we had earlier which is m x double dot plus m gamma x dot plus k x equal to 0.

So, the electrical mechanical analogy goes like x the count of part is q, the charge q in a capacitor the position x of the damped oscillator is like the charge q instantaneous charge q on the capacitor. The mass of this oscillator is the analog of the inductor, the gamma the friction constant here which has dimensions of time inverse is after you divide through by M or L as a case maybe this has the mentions of R over L.

Notice that L over R is the time constant of an LR circuit. So, we already see this analogy appearing and k the analog is 1 over C omega squared the analog is 1 over L C and you have exactly the same behavior for this LCR circuit with the resistor there as you have in this case and as before the condition for damping over damping etcetera. So, we want to have under damped circuit corresponds in the earlier case mechanical case to omega greater than gamma over 2 or in this case 1 over square root of L C greater than gamma was R over 2 L.

Or, so if I squares of both sides are 1 over LC greater than R squared 4 L squared. So, let us cancel out on L that is the condition, so and those conditions when we the fix the R L and C in such a way that the resistance is sufficient the small. So, the way to write this is to write this as R squared less than 4 L over C 2 squared root of L C. So, an LCR series circuit is under damped, if the resistance smaller than twice the square root of L over C.

In such a case you guarantee to have an approach that this charge of this capacitor is finally, going to tend to q equal to 0 q dot equal to 0 will happen in a oscillatory fashion. Such that, the charges go back and forth and oscillates will decreasing amplitude with an amplitude which is attenuated by this exponential factor control by the time constant L over R here L over R half of that and the fixes the way circuit behaves and of course, one could also look at the over damped and the critically damped cases.

If you recall when you want instrument to point towards which final reading, you would like no over oscillations, no over shooting. Because, then you takes time for it to you get back, it also not like over damping, because it takes the long, long time for it will reach oscillation. So, the most efficient way of arranging matters in such a case, when you want a meter to point to a final reading is as in the ballistic galvanometer in ancient use of an equipment is to make it critically damped. In other wards to fix matters such that this becomes in equality and when that happens you have the most efficient instrument for that particular application.

Now, there are many, many other cases where you use the damped simple harmonic oscillator. And this is not just a model of a damped oscillations, it is more than that it actually acts serves is the basic model of dissipation in a hues verity of applications of it is this is just one there many, many others. I should mention in the same breath that if you complicate the manner in which damp in occurs. For example, if assume in the mechanical case that the damped is proportional to the square of the velocity and not the velocity with appropriate sign.

The matter becomes immediately much more complicated or if you assume for instance that the friction is like dry friction and not discuss friction. So, there is a threshold over which it does not the access of frictional force and after that there is no frictional force at all one iteration threshold is crossed then you have yet another complication and the matter is a one again not so trivial at all. So, there are many, many complications which you can add on to this.

But, this simple damped harmonic oscillator serves as a basic paradigm of damped of dissipative motion of motion where the total energy is not conserve of oscillatory motion which slowly gets damped towards which is 0 and the under damped case in particular is a great significant in practical considerations and applications of the oscillator model.